Grouting in Rock and Concrete, Widmann (ed.) © 1993 Balkema, Rotterdam, ISBN 90 5410 350 7

An analytical solution of ground reaction curves for grouted tunnels

Eine analytische Lösung für einen injektionsverstärkten Tunnel mittels Bodenwiderstandskurven

Yanting Chang
Department of Soil and Rock Mechanics, Royal Institute of Technology, Stockholm, Sweden

Lars Hässler
Golder Geosystem AB, Uppsala, Sweden

ABSTRACT: Ground reaction curve concept is often used in tunnelling engineering for analysing interactions between rock mass and supports. Whereas there are already analytical solutions of ground reaction curves, treating the ground as a homogeneous medium, there is no such a solution for a grouted tunnel, where the ground has two layers of material with different properties. This paper presents an analytical solution, based on the theory of elasticity and theory of plasticity. The rock is considered as a perfectly-plastic medium after it yields, and the dilation of the rock mass is taken into account. This solution can be used not only for analysing grouted tunnels, but also for analysing tunnels with weakened zone caused by blasting. Computer programs of the solution are now available. Comparisons between the theory and numerical calculations are also presented.

ZUSAMMENFASSUNG: Bodenwiderstandskurven werden oft in der Tunnelbautechnik verwendet um die Zusammenwirkung zwischen dem Fels und der Verstärkung zu analysieren. Während es analytische Lösungen für Bodenwiderstandskurven im homogenen Untergrund gibt, fehlen solche Lösungen für den Fall wo der Tunnel von zwei Schichten mit verschiedenen Materialeigenschaften umgeben ist. In dem vorliegenden Aufsatz wird eine analytische Lösung unter Verwendung der Elastizitäts- und Plastizitätstheorie präsentiert. Der Fels wird im Bruchzustand als ideal-plastisches Material behandelt und die Dilatation der Felsmasse berücksichtigt. Die vorgeschlagene Lösung kann auch zur Analyse eines Tunnels mit einer durch Sprengung aufgelockerten Grenzschicht verwendet werden. Berechnungsprogramme zur Lösung des Problems sind vorhanden. Ein Vergleich mit numerischen Lösungen wird auch präsentiert.

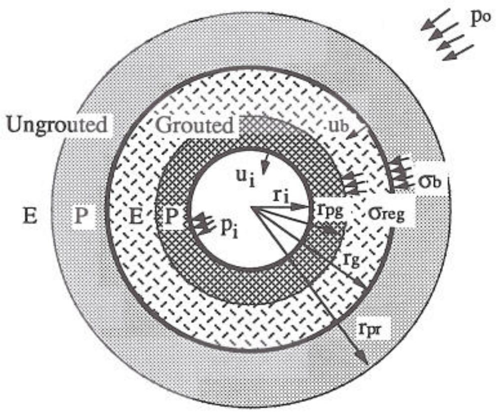
1. INTRODUCTION

Grouting is often used in tunnels in order to (i) tight the tunnel, (ii) enhance the strength of the rock mass. In accordance with the New Austrian Tunnelling Method (NATM), the Ground Reaction Curve concept (GRC) is often used to analyse the interaction between the rock mass and supports. Up to date, however, analytical solutions of the ground reaction curves are not available for grouted tunnels. This paper will present such an analytical solution, based on the theory of elasticity and theory of plasticity. The dilation of the rock mass is taken into account and the rock is considered as perfectly-plastic medium when it yields. Moreover, this solution is also applicable for tunnels with weakened zone caused by blasting.

The advantages of such solutions over numerical calculations are
i) it is easy to use,
ii) it is suitable for use *in-situ* because it gives immediate answer,
iii) it is convenient to make sensitivity studies of important parameters.

2. ANALYTICAL MODEL OF GROUTED TUNNEL

The model of a grouted tunnel to be analysed is shown in Figure 1. It is assumed that the tunnel is circular, the initial stress is hydro-static and the rock mass is grouted before tunnelling. At initial state ($p_i=p_o$), both grouted and ungrouted rock mass are elastic. When p_i is decreasing during tunnelling process, plasticity



r_i = radius of the tunnel,
r_g = radius of grouted zone,
r_{pg} = radius of the plastic zone in the grouted zone,
r_{pr} = radius of the possible plastic zone in the ungrouted rock,
p_i = internal pressure acting on the tunnel wall,
p_o = initial ground stress,
σ_b = radial contact stress at the boundary between the grouted and ungrouted rock,
σ_{reg} = radial stress at the elastic-plastic boundary in the grouted rock,
u_i = radius displacement of the tunnel wall,
u_b = radius displacement at the boundary between the grouted and ungrouted rock.

Fig.1 Analytical model of a grouted tunnel

Analytisches Modell eines injektierten Tunnels

may exist in either grouted zone or ungrouted zone or both zones. Because the grouted rock can be fully elastic or elasto-plastic or completely plastic, and the ungrouted rock can be fully elastic and elasto-plastic, there will be six cases that must be taken into account (see Table 1). In practice all of these cases are not likely to occur for a particular tunnel project. But all of them must be considered in the analytical solution.

In order to obtain the solution of the ground reaction curve (GRC) (i.e. the relation between p_i and u_i) of the grouted tunnel, the problem is treated as a perfect contact problem which can be described mathematically as

for ungrouted zone $u_{br} = f_1 (\sigma_{br}, p_o)$ (2.1)

Table 1 Six cases in the analytical solution

Case No	Grouted Rock	Ungrouted Rock	Symbol
1	Elastic	Elastic	E/E
2	Elastic	Elasto-Plastic	E/EP
3	Elasto-Plastic	Elastic	EP/E
4	Elasto-Plastic	Elasto-Plastic	EP/EP
5	Fully Plastic	Elastic	P/E
6	Fully Plastic	Elasto-Plastic	P/EP

for grouted zone: $\begin{cases} u_{bg} = f_2(\sigma_{bg}, p_i) \\ u_i = f_3(\sigma_{bg}, p_i) \end{cases}$ (2.2)

contact conditions: $\sigma_{br} = \sigma_{bg} = \sigma_b$ (2.3)
$u_{br} = u_{bg} = u_b$

where
u_{br} = displacement at the grout boundary from the ungrouted rock side,
u_{bg} = displacement at the grout boundary from the grouted rock side,
u_i = displacement of the tunnel wall,
s_{br} = radial stress acting on the ungrouted side at the grouted boundary,
s_{bg} = radial stress acting on the grouted side at the grouted boundary,
f_1 = a function describing the behaviour of the ungrouted rock,
f_2 and f_3 = functions describing the behaviour of the grouted rock.

Function f_1 for elastic and elasto-plastic conditions has been achieved by different authors, e.g. Stille (1983, 1989) and Hook & Brown (1980). While function f_2 and f_3 for elastic condition can be easily obtained from the theory of elasticity (e.g. Timoshenko et al, 1970), the solutions for elasto-plastic or fully plastic conditions have not been published up to present. When the functions f_1 and f_2 for all conditions are known, the contact stress σ_b can be solved by using the contact conditions (2.3), and then the displacement of the tunnel wall can be obtained by f_3 .

This paper will present a detailed derivation of function f_2 and f_3 for all the conditions of the grouted zone. For the convenience to the readers and the unity of the paper, the previous solutions of f_1 will also be given. Then solutions to all the six cases given in Table 1 will be presented.

3. SOLUTIONS FOR UNGROUTED ROCK

3.1 Elastic condition

By means of the plane strain elastic solution of a circular tunnel, equation

$$u_b = \frac{r_b}{2G}\left[p_o - \sigma_b\right] \tag{3.1}$$

holds for rock mass, where
G = shear modulus of ungrouted rock mass.

3.2 Elasto-plastic condition

For a perfect elasto-plastic material with the Mohr-Coulomb's failure criterion and a non-associated flow rule for the dilatancy after failure, the following solution for a circular tunnel in an infinite medium under hydrostatic initial ground pressure p_o will be achieved (Stille, 1983, 1989).

The deformation of the tunnel surface, u_i, is given as

$$\frac{u_b}{r_b} = \frac{A}{f+1}\left[2\left(\frac{r_p}{r_b}\right)^{f+1} + (f-1)\right] \tag{3.2}$$

$$\frac{r_p}{r_b} = \left[\frac{\sigma_{re} + a}{\sigma_b + a}\right]^{\frac{1}{k-1}} \tag{3.3}$$

where

r_p = radius of plastic zone in the ungrouted rock,

$$\sigma_{re} = \frac{2}{1+k}\left(p_o + a\right) - a \ ,$$

$$A = \frac{1+\mu}{E}\left(p_o - \sigma_{re}\right),$$

$$a = \frac{c}{\tan\phi},$$

$$k = \tan^2\left(45 + \frac{\phi}{2}\right)$$

c = cohesion of ungrouted rock mass,
ϕ = friction angle of ungrouted rock mass.
E = Young's modulus of ungrouted rock mass,
μ = Poisson's ratio of ungrouted rock mass,
f = the volume expansion after failure given by

$$f = \frac{\tan\left(45 + \frac{\phi}{2}\right)}{\tan\left(45 + \frac{\phi}{2} - \Psi\right)} \ ,$$

Ψ = dilatancy angle,

4. SOLUTIONS FOR GROUTED ROCK

4.1 Elastic condition

The grouted rock can be treated as a thick tube problem. Under plane strain condition, stresses in an elastic thick tube are given (Timoshenko, 1970)

$$\begin{Bmatrix}\sigma_r \\ \sigma_t\end{Bmatrix} = \begin{bmatrix} \frac{r_b^2 - r^2}{r_b^2 - r_i^2}\frac{r_i^2}{r^2} & \frac{r^2 - r_i^2}{r_b^2 - r_i^2}\frac{r_b^2}{r^2} \\ -\frac{r_b^2 + r^2}{r^2 - r^2}\frac{r_i^2}{r^2} & \frac{r^2 + r_i^2}{r^2 - r^2}\frac{r_b^2}{r^2} \end{bmatrix} \begin{Bmatrix} p_i \\ \sigma_b \end{Bmatrix} \tag{4.1}$$

From Hook's law for the plane strain condition, the total strains are given as

$$\begin{Bmatrix}\varepsilon_r \\ \varepsilon_t\end{Bmatrix} = \frac{1-\mu_g^2}{Eg}\begin{bmatrix} 1 & -\frac{\mu_g}{1-\mu_g} \\ -\frac{\mu_g}{1-\mu_g} & 1 \end{bmatrix}\begin{Bmatrix}\sigma_r \\ \sigma_t\end{Bmatrix} \tag{4.2}$$

where

E_g = elastic modulus of the grouted rock,
μ_g= Poisson's ratio of the grouted rock.

Because we are interested in the deformation caused by the excavation, the initial strain ε_o existing in the rock mass before the excavation should be subtracted from the total strains. The initial strain is determined by

$$\varepsilon_o = \frac{(1+\mu_g)(1-2\mu_g)}{E_g}p_o = -L\,p_o \tag{4.3}$$

$$L = -\frac{(1+\mu_g)(1-2\mu_g)}{E_g}$$

Therefore, from the compatibility conditions, the displacement caused by the excavation is expressed by

$$\frac{u}{r} = \varepsilon_t - \varepsilon_o \tag{4.4}$$

Substituting equations (4.1), (4.2) and (4.3) into (4.4) will give

$$\frac{u}{r} = M(r)\, p_i + N(r)\, \sigma_b + L\, p_o \qquad (4.5)$$

where

$$M(r) = \frac{1-\mu_g^2}{E_g}\left(-\frac{r_b^2+r^2}{r_b^2-r_i^2}\frac{r_i^2}{r^2} - \frac{\mu_g}{1-\mu_g}\frac{r_b^2-r^2}{r_b^2-r_i^2}\frac{r_i^2}{r^2}\right)$$

$$N(r) = \frac{1-\mu_g^2}{E_g}\left(\frac{r^2+r_i^2}{r_b^2-r_i^2}\frac{r_b^2}{r^2} - \frac{\mu_g}{1-\mu_g}\frac{r^2-r_i^2}{r_b^2-r_i^2}\frac{r_b^2}{r^2}\right)$$

When r is set to r_i, we will obtain the displacement of the tunnel wall

$$\frac{u_i}{r_i} = M(r_i)\, p_i + N(r_i)\, \sigma_b + L\, p_o \qquad (4.6)$$

and when $r = r_b$ the displacement at the boundary between the grouted and ungrouted rock is given by

$$\frac{u_b}{r_b} = M(r_b)\, p_i + N(r_b)\, \sigma_b + L\, p_o \qquad (4.7)$$

4.2 Elasto-plastic condition

When the grouted rock become plasticized, the rock is divided into two zones, the plastic zone and elastic zone. The radius of the plastic zone is noted as r_{pg} (refer to Figure 1).

From the compatibility conditions, the deformation in the plastic zone caused by excavation can be written as

$$\frac{u}{r} = \varepsilon_t\,, \qquad \frac{du}{dr} = \varepsilon_r \qquad (4.8)$$

where

ε_r, ε_t = relative radial and tangential strain in the plastic zone caused by excavation. These strains consist of a plastic part and an elastic part, i.e.

$$\varepsilon_t = \varepsilon_t^p + \varepsilon_t^e\,, \qquad \varepsilon_r = \varepsilon_r^p + \varepsilon_r^e \qquad (4.8a)$$

The non-associated flow law for the plastic strains is written as

$$\varepsilon_r^p = -f_g\, \varepsilon_t^p \qquad (4.9)$$

where f_g is the volume expansion factor after failure of the grouted rock. Combining Equation (4.8), (4.8a) and (4.9) will give following differential equation

$$\frac{du}{dr} + f_g \frac{u}{r} = \varepsilon_r^e + f_g\, \varepsilon_t^e \qquad (4.10)$$

If we assume that the elastic strains are constant in the plastic zone and equal to the value at the e-p boundary ($r = r_{pg}$), the solution of equation (4.10) can be easily obtained. Then the deformation of the tunnel wall is expressed as

$$\frac{u_i}{r_i} = \frac{1}{f_g+1}\left(\left(\varepsilon_t^e - \varepsilon_r^e\right)\left[\frac{r_{pg}}{r_i}\right]^{f_g+1} + \varepsilon_r^e + f_g\, \varepsilon_t^e\right) \qquad (4.11)$$

where the elastic strains can be determined by

$$\varepsilon_t^e = \frac{1-\mu_g^2}{E_g}\left(\sigma_{teg} - \frac{1-\mu_g}{\mu_g}\sigma_{reg}\right) + L\, p_o$$

$$\varepsilon_r^e = \frac{1-\mu_g^2}{E_g}\left(\sigma_{reg} - \frac{1-\mu_g}{\mu_g}\sigma_{teg}\right) + L\, p_o$$

$$\frac{\mu_g}{1-\mu_g}$$

and

$\sigma_{reg}, \sigma_{teg}$ = radial and tangential stress respectively at the e-p boundary, which will be determined later.

Assuming a perfect elasto-plastic material, the radius of the plastic zone r_{pg} in the grouted rock is given by

$$\frac{r_{pg}}{r_i} = \left(\frac{\sigma_{reg}+a_g}{p_i+a_g}\right)^{\frac{1}{k_g-1}} \qquad (4.12)$$

where

$$a_g = \frac{c_g}{\tan\phi_g}$$

$$k_g = \tan^2\left(45 + \frac{\phi_g}{2}\right)$$

c_g = cohesion of the grouted rock mass,
ϕ_g = friction angle of the grouted rock mass.

The displacement at the outer boundary of the grouted zone ($r = r_b$) is obtained by the compatibility condition and the fact that the outer part of the rock is elastic, e.g.

$$\frac{u_b}{r_b} = \varepsilon_{tb} + L\,p_o$$

$$= \frac{1-\mu_g^2}{E_g}\left(\sigma_{tb} - \frac{\mu_g}{1-\mu_g}\,\sigma_b\right) + L\,p_o \tag{4.13}$$

where

ε_{tb} = elastic tangential strain at $r = r_b$,
σ_{tb} = tangential stress at $r = r_b$,

Following is to find expressions for the stresses involved in calculating the displacements. The stresses in the elastic zone can be obtained by the same procedure as given the previous section.

$$\begin{Bmatrix}\sigma_r\\ \sigma_t\end{Bmatrix} = \begin{bmatrix} \dfrac{r_b^2 - r^2}{r_b^2 - r_{pg}^2}\dfrac{r_{pg}^2}{r^2} & \dfrac{r^2 - r_{pg}^2}{r_b^2 - r_{pg}^2}\dfrac{r_b^2}{r^2} \\ -\dfrac{r_b^2 + r^2}{r_b^2 - r_{pg}^2}\dfrac{r_{pg}^2}{r^2} & \dfrac{r^2 + r_{pg}^2}{r_b^2 - r_{pg}^2}\dfrac{r_b^2}{r^2} \end{bmatrix} \begin{Bmatrix}\sigma_{reg}\\ \sigma_b\end{Bmatrix} \tag{4.14}$$

From equation (4.14), the tangential stress σ_{tb} at $r = r_b$ is given

$$\sigma_{tb} = -\frac{2\,r_{pg}^2}{r_b^2 - r_{pg}^2}\,\sigma_{reg} + \frac{r_b^2 + r_{pg}^2}{r_b^2 - r_{pg}^2}\,\sigma_b \tag{4.15}$$

The tangential stress σ_{teg} at $r = r_{pg}$ can be written as

$$\sigma_{teg} = -\frac{r_b^2 + r_{pg}^2}{r_b^2 - r_{pg}^2}\,\sigma_{reg} + \frac{2\,r_b^2}{r_b^2 - r_{pg}^2}\,\sigma_b \tag{4.16}$$

It should be pointed out that σ_{reg} is a dependent variable of σ_b. In order to calculate σ_{tb} and σ_{teg} from equation (4.15) and (4.16), the relation between σ_{reg} and σ_b should be derived. Note that on the e-p boundary, the radial stress σ_{reg} and tangential stress σ_{teg} must meet the Mohr-Coulomb's criterion, i.e.

$$\frac{\sigma_{teg} - a_g}{\sigma_{reg} - a_g} = k_g \tag{4.17}$$

Eliminating σ_{teg} from equation (4.16) and (4.17) will give the relation

$$\sigma_{reg} = -\frac{(k_g - 1)(r_b^2 - r_e^2)a_g - 2r_b^2\sigma_b}{(r_b^2 - r_e^2)k_g + (r_b^2 + r_e^2)} \tag{4.18}$$

If the contact stress σ_b is known, all other stresses can be obtained from equation (4.18), (4.15) and (4.17). Then the elastic strains and the radius of the plastic zone can also calculated. Consequently the displacements u_i and u_b can be determined.

4.3 Completely plastic condition

The grouted zone will have no resistance to deformation when it becomes completely plastic, since the grouted rock is assumed to be a perfectly plastic material. In other words, the deformation of a completely plastic rock is arbitrary. However, the grouted rock in our case is bonded to the outer rock mass whose displacements u_b is determined by equation (3.1) or (3.2), i.e. the grouted rock is subjected to a defined displacement u_b. By using the compatibility conditions, the relation between u_b and u_i can be derived

$$u_i = u_b\left(\frac{r_b}{r_i}\right)^{f_g} + \frac{\varepsilon_r^e - f_g\,\varepsilon_t^e}{1+f_g}\left[r_i - \left(\frac{r_b}{r_i}\right)^{f_g} r_b\right] \tag{4.19}$$

where ε_r^e and ε_t^e are the elastic strains in the plastic grouted rock, which are assumed to be constant and equal to the value at r = rb, i.e.

$$\varepsilon_r^e = \frac{1-\mu^2}{E}\left(\sigma_b - \frac{\mu}{1-\mu}\sigma_{tb}\right) \tag{4.20a}$$

$$\varepsilon_t^e = \frac{1-\mu^2}{E}\left(\sigma_{tb} - \frac{\mu}{1-\mu}\sigma_b\right) \tag{4.20b}$$

where σ_b and σ_{tb} are determined by following equations

$$\sigma_b = (p_i + a_g)\left(\frac{r_b}{r_i}\right)^{k_g - 1} - a_g \tag{4.21a}$$

$$\sigma_{tb} = (p_i + a_g)\,k_g\left(\frac{r_b}{r_i}\right)^{k_g - 1} - a_g \tag{4.21b}$$

4.4 Solutions for the six cases of grouted tunnel

Based on the solutions given in the previous sections for grouted and ungrouted rock, we can obtain solutions for each particular case given in Table 1 (see Table 2). To determine which case the tunnel belongs to, the "tree

Table 2 Solutions for the cases

Case	Contact condition	u_i
E/E	Eq.(4.7) & (3.1)	(4.6)
E/EP	Eq.(4.7) & (3.2)	(4.6)
EP/E	Eq.(4.13) & (3.1)	(4.11)
EP/EP	Eq.(4.13) & (3.2)	(4.11)
P/E	u_b = (3.1)	(4.19)
P/EP	u_b = (3.2)	(4.19)

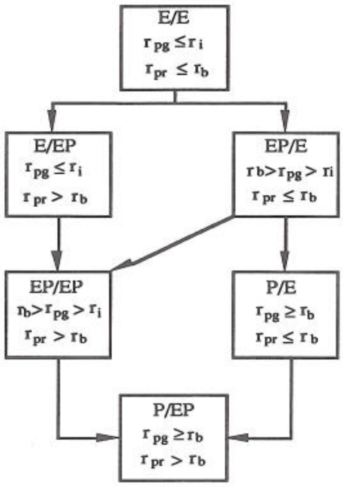


Fig. 2 The "tree approach" to determine the calculation flow

Die Verzweigungs-Methode zur Bestimmung des Berechnungsganges

approach" shown in Figure 2 is used in the computer program.

5. COMPARISON WITH NUMERICAL CALCULATIONS

In order to verity the formulation and the computer programs, ground reaction curves obtained from the solution are compared with that from numerical calculations by FLAC, version 3.22, for both tunnel with grouted zone and tunnel with weakened zone. Parameters used for the calculations and the properties of rock masses are given in Table 3 and Table 4 respectively. The mesh for the numerical model is shown in Figure 3. Results are shown in Figure 4 and Figure 5. It can be seen from the results that a good agreement is obtained between the analytical solution and numerical calculations.

Table 3 Parameters used in the calculations

Initial ground stress	1 MPa
Radius of tunnel	4.7m
Radius of grouted zone	7.7m
Radius of weakened zone	7.7m

Table 4 Properties of rock masses

	Tunnel with grouted zone		Tunnel with weakened zone	
	Origin rock	Grouted rock	Origin rock	Weak rock
E (GPa)	0.1	0.5	0.5	0.1
μ	0.2	0.2	0.2	0.2
ϕ	10°	15°	15°	10°
c (MPa)	0.1	0.2	0.2	0.1
ψ	10°	10°	10°	10 °

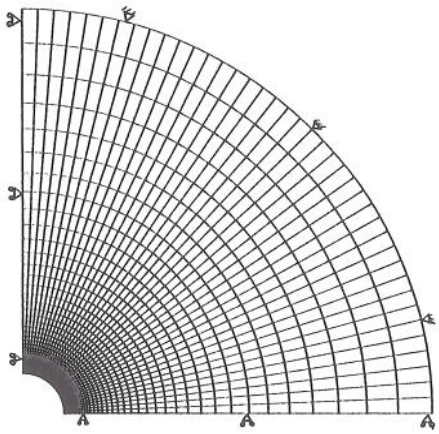


Fig.3 Mesh for numerical calculations

Netz für die numerische Berechnung

6. CONCLUSIONS

The analytical solution presented in this paper provides a useful tool for obtaining ground reaction curves for grouted tunnels. The advantage of such analytical solutions is that immediate answer can be obtained once the input data are available. Therefore it is very

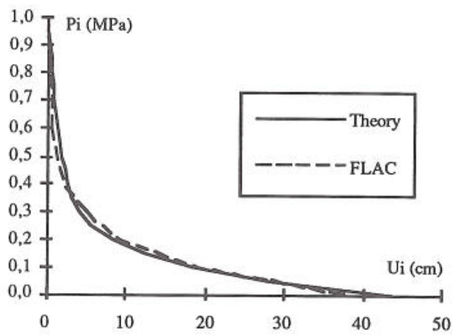


Fig. 4 Comparison between the theory and numerical results for tunnel with grouted zone

Vergleich zwischen der Theorie und dem numerischen Resultat für einen Tunnel mit einer injektierten Zone

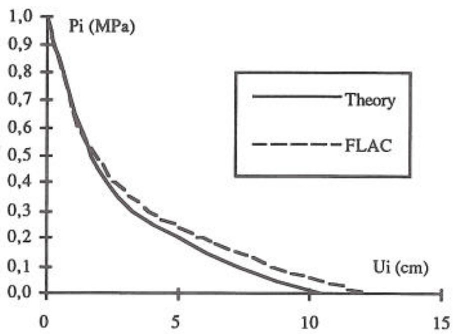


Fig. 5 Comparison between the theory and numerical results for tunnel with weakened zone

Vergleich zwichen der Theorie und dem numerischen Resultat für einen Tunnel mit einer geschwächten Zone

advantageous for the site engineers to use on a tunnel construction site.

The comparison between the results from the analytical solution and numerical calculations shows that a good agreement is obtained. This indicates that the formulation and computer program are reliable.

7. REFERENCES

CHANG, Y., 1992. "Influence of Early-Age Properties of Shotcrete on Tunnel Construction Sequences", Licentiate Thesis, Dept. of Soil and Rock Mechanics, the Royal Institute of Technology, Stockholm, Sweden.

HOEK, E., & BROWN, E. T., 1982. "Underground Excavation in Rocks", The Institute of Mining and Metallurgy.

STILLE, H., HOLMBERG, M. & NORD, G., 1989. "Support of Weak Rock with Grouted Bolts and Shotcrete", Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.,Vol.26, No.1, pp.99-113.

STILLE, H., 1983. "Theoretical Aspects on the Difference between prestressed Anchor Bolt and Grouted Bolt in Squeezing Rock", Proc. of the Int. Symp. on Rock Bolting, Abisko, 28 Aug.-2 Sept.

TIMOSHENKO, S. P ., GOODIER, J. N., 1970. "Theory of Elasticity", Mc Graw-Hill Book Company.

PROCEEDINGS OF THE INTERNATIONAL CONFERENCE ON GROUTING IN ROCK AND CONCRETE / SALZBURG / AUSTRIA / 11-12 OCTOBER 1993
BERICHTE DER INTERNATIONALEN KONFERENZ BETREFFEND INJEKTIONEN IN FELS UND BETON / SALZBURG / ÖSTERREICH / 11-12 OKTOBER 1993

Grouting in Rock and Concrete
Injizieren in Fels und Beton

Editor / Herausgeber
RICHARD WIDMANN
Österreichische Gesellschaft für Geomechanik

OFFPRINT

A.A. BALKEMA / ROTTERDAM / BROOKFIELD / 1993